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The Group of Units of a Simple Ring I*

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1. The structure of the group of units of a simple ring has been examined in special cases by Dieudonné [2], Rosenberg [6], and Kaplansky [4]. New methods for obtaining their results, and new ones, concerning the essential simplicity of the derived group of the group of units will appear in a future paper by the author. In the present work we are concerned with the solvability of normal subgroups of the group of units of a simple ring with nonidentity idempotent. We first show that noncentral normal subgroups are not solvable. This result is then expanded to subnormal subgroups and normal subgroups in semi-prime and prime rings.

2. All rings are assumed to be associative with an identity element. We will consistently denote rings by R or S, the group of units of R by U, and the center of R by Z. Note that Z is a field when R is simple. U' will denote the derived or commutator subgroup of U, and in general, $U^{(n)} = (U^{(n-1)})'$ the n-th derived subgroup. If H is a normal subgroup of G, we write $H \lhd G$. By a nonidentity idempotent e we mean that $e^2 = e$ and $e \neq 0, 1$.

DEFINITION 2.1. For units $u, v \in R$ let $(u, v) = u^{-1}v^{-1}uv$. For any $r, s \in R$ let $[r, s] = rs - sr$.

DEFINITION 2.2. If for some index set I, $1 = \sum_{i\in I} e_i$, where $\{e_i\}$ are orthogonal idempotents, then $R_{ij} = e_i Re_j$ and $e_i se_j = s_{ij}$.

LEMMA 2.3. *If R is a simple ring with $1 = \sum_{i\in I} e_i$, where $\{e_i\}$ are orthogonal idempotents, then $R_{ij} \neq 0$ and $R_{ij}R_{jk} = R_{ik}$.*

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Proof. Since R is a simple ring it is a prime ring, and so, if $aRb = 0$ we must have either $a = 0$ or $b = 0$. Thus if $R_{ij} = 0$, we have either $e_i = 0$ or $e_j = 0$. As idempotents are not zero, we must have $R_{ij} \neq 0$. By Definition 2.2 we have $R_{ij}R_{jk} = e_iRe_jRe_k$. The simplicity of R implies that $Re_jR = R$, so $R_{ij}R_{jk} = e_iRe_k = R_{ik}$.

3. We first turn to the case in which R contains two orthogonal idempotents whose sum is not the identity. That is $1 = e_1 + e_2 + e_3$ for the e_i orthogonal idempotents. The question of the solvability of U in this case is trivial and the result stronger than we can obtain for arbitrary normal subgroups of U.

LEMMA 3.1. *Let R be simple with $1 = \sum_{i \in I} e_i$ for orthogonal idempotents $\{e_i\}$ and card $I \geqslant 3$. If any subgroup G of the group of units U of R contains all units $1 + r_{ij}$ for $i, j \in I$ and $i \neq j$, then so does G', the derived group of G.*

Proof. Using the fact that for $i \neq j$ we have $(1 + r_{ij})^{-1} = 1 - r_{ij}$, we obtain $(1 + r_{ij}, 1 + r_{jk}) = 1 + r_{ij}r_{jk}$ for $i, j, k \in I$ and distinct. Since $(1 + s)(1 + r) = 1 + s + r$ for $s, r \in R_{ij}$ with $i \neq j$, we obtain $1 + r \in G'$ for $r \in R_{ik} = e_iRe_jRe_k$ for i and k arbitrary and distinct in I. Hence $1 + r_{ij} \in G'$ for $r_{ij} \in R_{ij}$ for all $i \neq j$.

THEOREM 3.2. *If R is simple with $1 = e_1 + e_2 + e_3$ with the e_i orthogonal idempotents, then U, the group of units of R, is not solvable.*

Proof. By Lemma 3.1 the group T generated by $1 + r_{ij}$ for $i \neq j$ with $i, j = 1, 2, 3$ is contained in each $U^{(k)}$ in the derived series of U. Hence $U^{(k)} \neq 1$ for any k. Thus the derived series of U does not terminate in the identity after a finite number of steps, and so U is not solvable.

The argument in Theorem 3.2 can clearly be extended transfinitely. That is, if we take some well ordered index set I, we can define for I a derived series [5, p. 182] for U which does not terminate in the identity. If $i \in I$ is not a limit ordinal let $U^{(i)} = (U^{(i-1)})'$, the derived group of $U^{(i-1)}$. If i is a limit ordinal, let $U^{(i)}$ be the derived group of $\bigcap_{j<i} U^{(j)}$. Then U will be transfinitely solvable if for some such I we have $\bigcap_{i \in I} U^{(i)} = 1$. In our case this cannot happen, so we have

COROLLARY 3.3. *With the hypothesis of Theorem 3.2, U is not transfinitely solvable.*

4. We turn now to the case where we assume only a single nonidentity idempotent. This case is considerably more involved and requires mild assumptions on the center of R. The important step is to generalize a theorem

of Amitsur. Before stating this theorem we state a theorem of Herstein which we will also require and which is used in the proof of Amitsur's theorem.

THEOREM 4.1. (Herstein) *If R is a simple ring and if $[T, [R, R]] \subset T$ for T an additive subgroup of R, then either $T \subset Z$ or $T \supset [R, R]$ except if R is of characteristic 2 and 4-dimensional over Z.*

Proof. [3, p. 17].

THEOREM 4.2. (Amitsur) *Let R be a simple ring whose centroid F is not $GF(2)$; suppose further that R has an idempotent $e \neq 0, 1$. Then any subspace of R invariant under all inner automorphisms of R must be 0, Z, or contain $[R, R]$. The only subalgebras of R invariant under all inner automorphisms of R are 0, Z, and R.*

Proof. [1, p. 988–989].

COROLLARY 4.3. *Let R be simple and contain a nonidentity idempotent e. If N is an Abelian subgroup of U, and $N \lhd U$, then $N \subset Z$, provided $Z \neq GF(2)$.*

Proof. The ring S generated by N and Z is clearly invariant with respect to conjugation by all units of U. Since $Z \neq GF(2)$, we have by Theorem 4.2 that $S = Z$ or $S = R$. But $S = Z$ implies that $N \subset Z$ while $S = R$ says that R is commutative, contradicting the existence of e.

Let us consider a noncentral normal subgroup N. If N is solvable then we have $N \supset N' \supset \cdots \supset N^{(k)} \supset 1$. Now $N^{(k)}$ is an Abelian normal subgroup of U and so is in Z, if $Z \neq GF(2)$, by Corollary 4.3. If we could show that $N^{(k-1)}$ is also in Z we would have a contradiction, and so, N could not be solvable. It is with this aim in mind that we want to generalize Theorem 4.2.

LEMMA 4.4. *Let R be a simple ring and W a subspace of R, over Z. Suppose that for some $a \in R$ we have n functions $f^i(za)$ $z \in Z$, such that $f^i(za) = z^i f^i(a)$ and $f^1(za) + \cdots + f^n(za)$ is in W. Then if* card $Z \geqslant n + 1$ *we have $f^i(za) \in W$ for each i.*

Proof. As card $Z \geqslant n + 1$ we can find $z \in Z$ with $z^2 - z \neq 0$. Then both $f^1(za) + \cdots + f^n(za)$ and $z^2(f^1(a) + \cdots + f^n(a))$ are in W. Hence the difference $(z^2 - z)f^1(a) + (z^2 - z^3)f^3(a) + \cdots + (z^2 - z^n)f^n(a)$ is in W. Now choose $z_1 \in Z$ with $z_1^3 - z_1 \neq 0$ and use it to eliminate the $f^3(a)$ term from the above sum. Continuing we obtain $f^1(za)$ in W. A repetition of the argument on $f^2(za) + \cdots + f^n(za)$ yields $f^2(za)$ in W and so on until we have $f^i(za)$ in W for each i.

THEOREM 4.5. *Let R be a simple ring containing an idempotent $e \neq 0, 1$ and suppose the center Z of R has at least five elements. If W is a subspace of R*

invariant under all inner automorphisms by elements of $N \lhd U$ with $N \not\subset Z$, then W is 0, Z, or contains $[R, R]$. If W is a subalgebra of R then W is 0, Z, or R.

Proof. For any $g \in N$ and $a \in R$ with $a^2 = 0$, we have $(1 + a, g)$ in N. So for $w \in W$ we have $(1 + a, g)w(g, 1 + a) \in W$. That is,

$$(1 - a) g^{-1}(1 + a) gwg^{-1}(1 - a) g(1 + a) \in W.$$

Expanding we obtain

$$\begin{aligned} &w + (wa - aw + g^{-1}agw - wg^{-1}ag) \\ &\quad + (g^{-1}agwa - ag^{-1}agw - awa + awg^{-1}ag - g^{-1}agwg^{-1}ag - wg^{-1}aga) \\ &\quad + (awg^{-1}aga - ag^{-1}agwa + ag^{-1}agwg^{-1}ag - g^{-1}agwg^{-1}aga) \\ &\quad + (ag^{-1}agwg^{-1}ag) \in W. \end{aligned}$$

Now for $z \in Z$ and $a^2 = 0$ we have $(za)^2 = 0$ and so the above sum, with za replacing "a" is also in W. Clearly, each expression in parentheses in the above sum is an $f^i(za)$ as in Lemma 4.4. Hence we have

$$w + f^1(za) + \cdots + f^4(za) \in W,$$

where

$$\begin{aligned} f^1(a) &= (wa - aw + g^{-1}agw - wg^{-1}ag) \\ f^2(a) &= (g^{-1}agwa - ag^{-1}agw - awa + awg^{-1}ag - wg^{-1}aga - g^{-1}agwg^{-1}ag) \\ f^3(a) &= (awg^{-1}aga - ag^{-1}agwa + ag^{-1}agwg^{-1}ag - g^{-1}agwg^{-1}aga) \\ f^4(a) &= (ag^{-1}agwg^{-1}aga). \end{aligned}$$

Since W is a subspace, we can subtract w to obtain

$$f^1(za) + f^2(za) + f^3(za) + f^4(za) \in W.$$

Since Z has at least five elements, we can invoke Lemma 4.4 to obtain $f^1(a) \in W$. We re-write this as $[w, a - g^{-1}ag] \in W$. Let T be the additive subgroup of R generated by all $a - g^{-1}ag$ for $a^2 = 0$ and g in N. Clearly T is a subspace of R. Further, if u is a unit in R then

$$u^{-1}(a - g^{-1}ag)u = u^{-1}au - (u^{-1}g^{-1}u)(u^{-1}au)(u^{-1}gu)$$

and note that $(u^{-1}au)^2 = 0$ and $u^{-1}gu \in N$. Thus T is invariant under all inner automorphisms of R and so, by Theorem 4.2 T is either $0, Z$, or contains $[R, R]$. If $T = 0$, then each $g \in N$ centralizes every $a \in R$ with $a^2 = 0$. In particular, N centralizes $eR(1 - e)$ and $(1 - e) Re$. Since R is simple,

$$eRe = eR(1 - e) Re \qquad \text{and} \qquad (1 - e)R(1 - e) = (1 - e)ReR(1 - e),$$

so $N \subset Z$, a contradiction. If $T = Z$ we have, for some $a^2 = 0$ and $g \in N$, that $a - g^{-1}ag \in Z$ and is not zero. Then $a(a - g^{-1}ag) = (a - g^{-1}ag)a$ and so $ag^{-1}ag = g^{-1}aga$. Hence $(a - g^{-1}ag)^2 = -2ag^{-1}ag$ must be a nonzero element of Z which is a zero divisor. This is not possible. Thus the last possibility, that $[R, R] \subset T$ must hold. But now, as W is a subspace of R, and since $[W, T] \subset W$, we have that $[W, [R, R]] \subset W$, so by Theorem 4.1 W is 0, Z, or contains $[R, R]$ unless R is of characteristic 2 and is 4-dimensional over Z. Assuming for the moment the truth of the theorem in this special case, the first statement of the theorem is proved. The second follows from the fact that the subring generated by $[R, R]$ is R [3, p. 9].

It remains to prove that if R is of characteristic 2 and is 4-dimensional over Z, and if W is a subspace of R invariant with respect to conjugation by elements of a normal subgroup N of U, and such that $[W, [R, R]] \subset W$, then W is 0, Z, or contains $[R, R]$. At this point, since we are assuming that Z contains at least five elements, we could use a theorem of Dieudonné [2, *Théoremè* 3] and Theorem 4.2 to conclude that if $N \not\subset Z$ then $U' \subset N$. If Z^* denotes the nonzero elements of Z, then $U = Z^*U'$ and so W is invariant with respect to conjugation by every unit of R. We could now obtain our result by using Theorem 4.2. We prefer to handle this special case, however, without making use of the theorem of Dieudonné.

Since we are assuming that R is simple and 4-dimensional over its center, the theory of rings with minimum condition tells us that in the case we are considering $R = Z_2$, the 2×2 matrix ring over Z. Hence R contains the usual matrix units e_{11}, e_{12}, e_{21}, and e_{22}. Therefore, $[e_{11}, e_{12}] = e_{12}$ and $[e_{22}, e_{21}] = e_{21}$ are in $[R, R]$. Let $w = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be in W. As $[W, [R, R]] \subset W$ we have $[w, e_{12}] = \left(\begin{smallmatrix} c & a+d \\ 0 & c \end{smallmatrix}\right)$ and $[[w, e_{12}], e_{21}] = \left(\begin{smallmatrix} a+d & 0 \\ 2c & a+d \end{smallmatrix}\right)$ both in W. Since the characteristic of R, and so Z, is 2, $2c = 0$ and it follows, since W is a subspace, that $1 \in W$ unless $a + d = c = 0$. So if $1 \notin W$ we must have a typical element of W be of the form $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ with $b \neq 0$. But as we have seen, commutation with e_{21} will imply that $1 \in W$. Hence $1 \in W$ and so $\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) \in W$ for $c \in Z$, unless $W = 0$. Assume now that $W \neq 0, Z$. We want $[R, R] \subset W$.

If $w = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in W and $a \neq d$, then as above we can obtain $\left(\begin{smallmatrix} c & a+d \\ 0 & c \end{smallmatrix}\right) \in W$ which implies that $(a + d)\, e_{12}$, and so e_{12}, is in W. Similarly $e_{21} \in W$. A direct computation shows that as a subspace of R, $[R, R]$ is generated by 1, e_{12}, e_{21} over Z. Hence $W \supset [R, R]$ unless $w \in W$ is of the form $\left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right)$. Let us henceforth assume that $W \not\supset [R, R]$. Since $1 \in W$ and we are assuming that $W \not\subset Z$, it follows that $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in W$, with $b, c \neq 0$. As W is a subspace $\left(\begin{smallmatrix} 0 & 1 \\ k & 0 \end{smallmatrix}\right) \in W$ for some nonzero k in Z. A simple argument now shows that a typical $w = \left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right)$ in W is of the form $\left(\begin{smallmatrix} a & b \\ bk & a \end{smallmatrix}\right)$ for $k \neq 0$ in Z. We now examine N.

Suppose that $\left(\begin{smallmatrix} x & y \\ w & t \end{smallmatrix}\right)$ is a unit in R and conjugation by it leaves W invariant. Note that its inverse is $z\left(\begin{smallmatrix} t & y \\ w & x \end{smallmatrix}\right)$ where $z \in Z$ and non-zero. Now this says that $\left(\begin{smallmatrix} t & y \\ w & x \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ bk & a \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ w & t \end{smallmatrix}\right) \in W$, and suppose $b \neq 0$. Multiplying out and looking at the standard form for an element of W yields the condition

$$y^2k^2 + (x^2 + t^2)\, k + w^2 = 0. \tag{A}$$

In particular, any element of N must satisfy (A). Let $\begin{pmatrix} x & y \\ w & t \end{pmatrix} \in N$. Since $N \lhd U$ we have $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ w & t \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in N$. That is, $\begin{pmatrix} x+w & x+y+w+t \\ w & w+t \end{pmatrix} \in N$. Using condition (A) we obtain both

$$y^2k^2 + (x^2 + t^2)\,k + w^2 \quad \text{and} \quad (x^2 + y^2 + w^2 + t^2)\,k^2 + (x^2 + t^2)\,k + w^2$$

equal zero. Adding, we get $0 = x^2 + w^2 + t^2 = (x + w + t)^2$, which implies $x + w + t = 0$. Similarly we get $x + t + y = 0$ by using $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ w & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in N$. Thus we obtain $w + y = 0$, or $w = y$. So $v \in N$ implies that $v = \begin{pmatrix} x & y \\ y & t \end{pmatrix}$.

Now consider $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ y & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in N$. Expanding gives $\begin{pmatrix} y+t & x+t \\ y & y+x \end{pmatrix} \in N$ and so $y = x + t$. Thus if $v \in N$ we must have $v = \begin{pmatrix} x & x+t \\ x+t & t \end{pmatrix}$. If $\begin{pmatrix} r & s \\ u & p \end{pmatrix}$ is a typical unit of R with inverse $z\begin{pmatrix} p & s \\ u & r \end{pmatrix}$, then we have $z\begin{pmatrix} r & s \\ u & p \end{pmatrix}\begin{pmatrix} x & x+t \\ x+t & t \end{pmatrix}\begin{pmatrix} p & s \\ u & r \end{pmatrix} \in N$, where we assume $x + t \neq 0$ for some element of N. Equating the off diagonal elements gives $rs + s^2 + r^2 = up + u^2 + p^2$ for every unit in R. But this is impossible if $Z \neq GF(2)$, for pick $z \in Z$ with $z \neq z^3$. The unit $\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}$ does not satisfy the necessary condition. Hence our assumption that $[R, R] \not\subset W$ leads us to a contradiction, and so we must conclude that $[R, R] \subset W$, if $W \neq 0$, or Z.

Definition 4.6. Given $S \subset R$ let $N(S) = \{r \in R \mid \text{for each } s \in S,\ rs = s_1 r \text{ with } s_1 \in S\}$ and $C(S) = \{r \in R \mid rs = sr \text{ for all } s \in S\}$. $N(S)$ is called the normalizer of S in R and $C(S)$ is called the centralizer of S in R. If $T \subset R$ then $N(S) \cap T = N_T(S)$ and $C(S) \cap T = C_T(S)$.

In the corollaries that follow, R will denote a simple ring with nonidentity idempotent and with card $Z \geqslant 5$.

Corollary 4.7. *Let $s \in R$ $s \notin Z$. Then if $Z[s]$ is the ring of polynomials in s over Z, $N(Z[s])$ contains no noncentral normal subgroups of U.*

Proof. Suppose $G \subset N(Z[s])$ is a noncentral normal subgroup of U. Then $Z[s]$ is invariant with respect to conjugation by elements of G. By Theorem 4.5, $Z[s]$ is 0, Z, or R. As $s \notin Z$ we must have $Z[s] = R$. But then R is commutative, contradicting the existence of a nonidentity idempotent. Hence $G \subset Z$ or is not normal.

Corollary 4.8. *Given $N_1, N_2 \lhd U$, $N_1, N_2 \not\subset Z$ and (N_1, N_2) the commutator of N_1 and N_2 (that is, the group generated by (n_1, n_2) for $n_i \in N_i$), then $(N_1, N_2) \not\subset Z$ and so $N_1 \cap N_2 \not\subset Z$.*

Proof. If $(N_1, N_2) \subset Z$ then there exists an $n_1 \in N_1$, $n_1 \notin Z$ with $n_1^{-1}n_2^{-1}n_1n_2 \in Z$ for all $n_2 \in N_2$. But then $N_2 \subset N(Z[n_1])$ contradicting N_2 normal and not central, by Corollary 4.7. Since $(N_1, N_2) \subset N_1 \cap N_2$, it follows that $N_1 \cap N_2 \not\subset Z$.

COROLLARY 4.9. *If $N \lhd U$, $N \not\subset Z$, then N is not solvable.*

Proof. If N is solvable consider its derived series:

$$N \supset N' \supset \cdots \supset N^{(k)} \supset N^{(k+1)} = 1.$$

Now $N^{(k)}$ is an Abelian normal subgroup of U, and so, $N^{(k)} \subset Z$ by Corollary 4.3. But then $N^{(k-1)}$ is a normal subgroup of U and $(N^{(k-1)}, N^{(k-1)})$ is in Z, so by Corollary 4.8 $N^{(k-1)} \subset Z$. This contradiction shows that N is not solvable.

5. One might ask whether the result in Corollary 4.9 holds true in a simple ring with nonidentity idempotent where Z may be the field of two, three, or four elements. In the case of $Z = GF(2)$ or $GF(3)$, consideration of the 2×2 matrix ring over these fields shows that the result cannot hold in general. We will now show that if $Z = GF(4)$ or if $1 = e_1 + e_2 + e_3$, for the e_i orthogonal idempotents, and $Z = GF(3)$, then the result of Corollary 4.9 holds. Our method will be to prove what we need of Corollary 4.8 in each case. For then the proof of Corollary 4.9 will hold and we will be done.

Assume that $Z = GF(3)$ and $1 = e_1 + e_2 + e_3$ for the e_i orthogonal idempotents. Let N be a noncentral, solvable, normal subgroup of U with derived series $N \supset N' \supset \cdots \supset N^{(k)} \supset 1$. By Corollary 4.3 $N^{(k)} \subset Z$. Thus if $x, y \in N^{(k-1)}$, $xy = zyx$, where $z = 1, -1$ since $Z = GF(3)$. It follows that $x^2y = yx^2$ and x^2 is in the center of $N^{(k-1)}$, a normal Abelian subgroup of U. Hence $x^2 \in Z$. Now if $a \in R$ and $a^2 = 0$, then for $x \in N^{(k-1)}$, $(1 - a)x(1 + a) \in N^{(k-1)}$ and we have that

$$(1 - a)x(1 + a)x = zx(1 - a)x(1 + a).$$

Suppose that $(1 - a)x(1 + a)x = -x(1 - a)x(1 + a)$. Expanding, and noting that $x^2 \in Z$, we get $2x^2 = xaxa + axax \in Z$. If we assume that $a \in R_{ij}$ for $i \neq j$ and pick $k \neq i, j$, then since $e_k a = 0$ and $2x^2 \neq 0 \in Z$, we must have $e_k xaxa \neq 0$. But then $0 = e_k(2x^2)\, e_j = e_k xaxae_j = e_k xaxa$. This contradiction establishes that for $a \in R_{ij}$ we must have

$$(1 - a)x(1 + a)x = x(1 - a)x(1 + a).$$

Expanding this gives $y = 2ax^2 - 2xax - xaxa + axax = 0$. Since $0 = aya = -2axaxa$, we have $axaxa = 0$. As $0 = ay$, we get $0 = -2axax - axaxa = -2axax$, and so $axax = 0$. Also

$$0 = ya = (2ax^2 - 2xax - xaxa)a = 2ax^2a - 2xaxa = -2xaxa$$

since $x^2 \in Z$. It follows that $0 = y = 2ax^2 - 2xax$, and so $ax^2 = xax$. Since x is a unit we have $ax = xa$. This says that if $x \in N^{(k-1)}$, then x commutes with R_{ij} for $i \neq j$. But since $R_{kk} = R_{ki}R_{ik}$, by Lemma 2.3, we have that $N^{(k-1)} \subset Z$. This establishes

THEOREM 5.1. *Let R be simple with* $1 = e_1 + e_2 + e_3$, *for the* e_i *orthogonal idempotents, and let* $Z = GF(3)$. *Then no noncentral normal subgroup of U is solvable.*

We return to the situation in which R contains a nonidentity idempotent, and now assume that $Z = GF(4)$. As in the proof of Theorem 5.1 we consider a noncentral, solvable, normal subgroup N and its derived series. As above we get $N^{(k)} \subset Z$ and for $x \in N^{(k-1)}$ we get $x^3 \in Z$. Pick $a \in R$ with $a^2 = 0$. Then $y = (1 + a)^{-1}x(1 + a)\,x^{-1} \in N^{(k-1)}$ for $x \in N^{(k-1)}$, and $y^3 \in Z$. Since $Z = GF(4)$, $(1 + a)^{-1} = (1 + a)$. Using this in the expansion of y^3 gives

$$y^3 = 1 + a + xax^{-1} + xax^{-1}a + (xax^{-1}a)^2 + (axax^{-1})^2 + (axax^{-1})^2a$$
$$+ xax^{-1}(axax^{-1})^2 + (axax^{-1})^3.$$

Now $0 = ay^3a = axax^{-1}a + (axax^{-1})^3a$. Hence $(axax^{-1})^2 = (axax^{-1})^4$ for any $x \in N^{(k-1)}$ and $a \in R$ with $a^2 = 0$. Let $z \in Z$. Then $((za)x(za)\,x^{-1})^2 = ((za)x(za)\,x^{-1})^4$, which implies that $z^4(axax^{-1})^2 = z^8(axax^{-1})^4$. Since $z = GF(4)$, $z^4 = z$ for $z \in Z$ and there is $z \in Z$ with $z^2 \neq z$. By the usual methods of elimination we can obtain $(axax^{-1})^2 = 0$. Using the expression for y^3 and the fact that $(axax^{-1})^2 = 0$, we obtain

$$t = 1 + y^3 = a + xax^{-1} + xax^{-1}a + (xax^{-1}a)^2$$

is in Z. Clearly, $0 = ata = axax^{-1}a$. and so $a + xax^{-1} + xax^{-1}a$ is in Z. Since $(za)^2 = 0$ for $z \in Z$, and since Z contains more than two elements, if we let $W = Z$, $f^1(za) = za + xzax^{-1}$, and $f^2(za) = x(za)\,x^{-1}(za)$, we can apply Lemma 4.4 to obtain $xax^{-1}a \in Z$. As $xax^{-1}a$ is a zero divisor in the center of a simple ring, we must have $0 = xax^{-1}a$ and so $axa = 0$ for $x \in N^{(k-1)}$ and $a^2 = 0$. Let S be the subspace generated by $N^{(k-1)}$ over Z. As $N^{(k-1)} \not\subset Z$, and as S is a subalgebra of R invariant under all inner automorphisms of R, we have, by Theorem 4.2 that $S = R$. Hence $aRa = 0$ for $a^2 = 0$. But as R is a simple ring it is prime, and so $a = 0$, while there do exist nonzero elements whose square is zero, namely elements of $eR(1 - e)$. This contradiction leads us to conclude that $N^{(k-1)} \subset Z$. Thus N cannot be solvable, and we have

THEOREM 5.2. *Let R be a simple ring with nonidentity idempotent and let* $Z = GF(4)$. *Then no noncentral normal subgroup of U is solvable.*

6. Knowing the situation for normal subgroups of U, it is natural to try and determine whether any group in a normal chain is solvable. In fact, does Theorem 4.5 hold for such subgroups? Our methods work, with a little less generality than in section four, for normal subgroups of normal subgroups, which we call *subnormal subgroups*.

We now state a result of Levitzki which we will need in what follows.

LEMMA 6.1. *Let R be a ring and J a non-zero right (left) ideal of R. Suppose that given $a \in J$, $a^n = 0$ for a fixed integer n; then R contains a nonzero nilpotent ideal.*

Proof. [3, p. 1].

LEMMA 6.2. *Suppose that R has no nonzero nilpotent ideals and contains a non-identity idempotent. If some $s \in R$ satisfies $(as)^k a = 0$ for all $a \in R$ with $a^2 = 0$ and with k a fixed integer, then s commutes with $eR(1 - e)$ for any idempotent $e \in R$. If R is simple then $s \in Z$.*

Proof. Let e be a nonidentity idempotent in R. Clearly every element of $eR(1 - e)$ has square zero. Thus if $a \in eR(1 - e)$ we have $(as)^k a = 0$. Hence for $a = er(1 - e)$ we obtain $(r(1 - e)\, se)^{k+2} = 0$. So $R(1 - e)\, se$ is a nil left ideal of index $k + 2$, and so by Lemma 6.1 R contains a nonzero nilpotent ideal unless $R(1 - e)\, se = 0$. By our hypothesis we must have $R(1 - e)\, se = 0$. Now $T = \{t \in R \mid Rt = 0\}$ is a nilpotent ideal of R and so is zero. Thus $(1 - e)\, se = 0$. Repeating the argument for $a \in (1 - e)\, Re$, we get $es(1 - e) = 0$. Hence s commutes with all idempotents of R. It is trivial to verify that if e is an idempotent of R then so is $f = e - er(1 - e)$. Since s commutes with e, it now follows that s commutes with $eR(1 - e)$. If R is simple then by Lemma 2.3 $eRe = eR(1 - e)\, Re$ and $(1 - e)\, R(1 - e) = (1 - e)\, ReR(1 - e)$, so $s \in Z$.

THEOREM 6.3. *Let R be a simple ring with nonidentity idempotent, and suppose that* card $Z \geqslant 9$ *and that R is not 4-dimensional over Z if Z has characteristic 2. If W is a subspace of R invariant with respect to conjugation by elements of $G \lhd N \lhd U$ with $G \not\subset Z$, then W is 0, Z, or $W \supset [R, R]$. If W is a subalgebra, then W is 0, Z, or R.*

Proof. We proceed much as in Theorem 4.5. For any $a \in R$ with $a^2 = 0$, and for $g \in G \subset N$, we have $(1 - a)g(1 + a) \in N$. Since $G \lhd N$, $h = ((1 - a)g^{-1}(1 + a), g) = (1 - a)g(1 + a)g^{-1}(1 - a)g^{-1}(1 + a)g$ is in G. Thus if $w \in W$ we get $hwh^{-1} \in W$. We expand this expression to obtain a sum of terms, as in Theorem 4.5, which we rearrange, collecting all those together which contain "a" the same number of times. For $z \in Z$, $(za)^2 = 0$, so we could have begun with za instead of a. Thus what we have, after expanding hwh^{-1}, is an expression $w + f^1(za) + \cdots + f^8(za)$ in W, where $f^i(za) = z^i f^i(a)$. As W is a subspace and $w \in W$, it follows that

$f^1(za) + \cdots + f^8(za) \in W$. Since Z has at least nine elements, we can apply Lemma 4.4 to conclude that $f^1(a) = [w, 2a - gag^{-1} - g^{-1}ag] \in W$.

Let T be the additive subgroup of R generated by all $2a - gag^{-1} - g^{-1}ag$ for $a^2 = 0$ and $g \in G$. Clearly T is a subspace of R, and since $G \lhd N$, T is invariant with respect to conjugation by elements of N. This follows by conjugating a typical generator of T by $v \in N$ and noting that $(vav^{-1})^2 = 0$ and $vgv^{-1} \in G$. Thus Theorem 4.5 applies and we can conclude that T is 0, Z, or $T \supset [R, R]$. If $T \supset [R, R]$ we are done exactly as in Theorem 4.5. That is, we would have $[W, [R, R]] \subset W$ and Theorem 4.1 would imply that $W \subset Z$ or $W \supset [R, R]$. If W were a subalgebra of R, the fact that the ring generated by $[R, R]$ is R [3, p. 9] would give the second conclusion of the theorem. Hence we may assume that $T \not\supset [R, R]$ which means that $T \subset Z$. We proceed by considering two cases.

Case I. char $Z \neq 2$. Let $2a - gag^{-1} - g^{-1}ag = z \in Z$. Now $zaga = -gag^{-1}aga - g^{-1}agaga$, and so $zaga + gag^{-1}aga + g^{-1}agaga = 0$. Clearly, we could replace each "a" in this expression by ta for $t \in Z$. Let $f^1(ta) = 0$, $f^2(ta) = z(ta)g(ta)$, and $f^3(ta) = g(ta)g^{-1}(ta)g(ta) + g^{-1}(ta)g(ta)g(ta)$. As $f^i(ta) = t^i f^i(a)$, if we let W in Lemma 4.4 be zero, we can apply Lemma 4.4 to our situation, as card $Z > 3$, to obtain $f^2(a) = 0$. That is $zaga = 0$. If $z \neq 0$, then $aga = 0$. If $z = 0$, then $0 = agzga = 2agaga$. As the characteristic of Z is not 2, it follows that $agaga = 0$. Thus in either case $agaga = 0$ for any $g \in G$ and $a^2 = 0$. By Lemma 6.2 $g \in Z$, and so $G \subset Z$, a contradiction. Hence for char $Z \neq 2$ we must have $T \supset [R, R]$.

Case II. char $Z = 2$. Recall that we are assuming that $T \subset Z$. Now, for $g \in G$ and $a^2 = 0$, we have $gag^{-1} + g^{-1}ag = z \in Z$. If $z \neq 0$ we get $aga = 0$, exactly as in Case I. If $z = 0$ then $gag^{-1} = g^{-1}ag$ and it follows that $g^2a = ag^2$. Thus $ag^2a = aag^2 = 0$. So in either situation $ag^2a = 0$. By Lemma 6.2, $g^2 \in Z$.

As $G \lhd N \lhd U$, for $a^2 = 0$ and $g \in G$, $(1 + a)g(1 + a) \in N$, and $h = (1 + a)g(1 + a)g(1 + a)g^{-1}(1 + a) \in G$. Then $hg \in G$, and so $(hg)^2 \in Z$. That is $((1 + a)g)^2(1 + a)g^{-1}((1 + a)g)^3(1 + a)g^{-1}(1 + a)\,g \in Z$. Using the well-known fact that for units u and v, if $uv \in Z$ then $uv = vu$, we have $k = ((1 + a)g)^3(1 + a)\,g^{-1}((1 + a)g)^3(1 + a)\,g^{-1} \in Z$. Since $g^2 \in Z$, $kg^2 \in Z$, and this says that $((1 + a)g)^3(1 + a)\,g^{-1}((1 + a)g)^4 \in Z$. Thus

$$((1 + a)g)^7(1 + a)\,g^{-1} \in Z.$$

Again using $g^2 \in Z$, we finally obtain $((1 + a)g)^8 \in Z$. Clearly we could replace "a" by za for $z \in Z$. If we expand $((1 + za)g)^8$ and collect terms in the usual way, by occurences of za, we get an expression $g^8 + f^1(za) + \cdots + f^8(za) \in Z$, where $f^i(za) = z^i f^i(a)$ and $f^8(a) = (ag)^8$. Since $g^8 \in Z$ and Z contains at least nine elements, we can use Lemma 4.4 on $f^1(za) + \cdots + f^8(za)$ with W in the lemma equal to Z, to get $f^8(a) = (ag)^8 \in Z$. It follows that $(ag)^8 a = a(ag)^8 = 0$. Thus Lemma 6.2 applies, and as in Case I we must conclude that $T \supset [R, R]$.

The corollaries following Theorem 4.5 have their corresponding statements for subnormal subgroups. We state them now for completeness, in a single corollary. All of the proofs, except one, are omitted because of their similarity to those of the earlier corollaries.

COROLLARY 6.4. *Let R be simple with nonidentity idempotent and suppose that* card $Z \geqslant 9$, *and that R is not 4-dimensional over Z if Z is of characteristic* 2. *Then*

(i) *For $s \in R$, $s \notin Z$, $N(Z[s])$ contains no non-central subnormal subgroup of U.*

(ii) *If $G \lhd N \lhd U$, then $G \subset Z$ or $C_N(G) \subset Z$.*

(iii) *If $G \lhd N \lhd U$ and G is Abelian, then $G \subset Z$.*

(iv) *Given G_1, $G_2 \lhd N \lhd U$ with G_1, $G_2 \not\subset Z$ and (G_1, G_2) the commutator of G_1 and G_2, then $(G_1, G_2) \not\subset Z$ and so $G_1 \cap G_2 \not\subset Z$.*

(v) *If $G \lhd N \lhd U$ and $G \not\subset Z$, then G is not solvable.*

Proof of (ii). If $G \not\subset Z$ then a noncentral element $s \in G$ has $C_N(G)$ in its centralizer. That is $C_N(G) \subset N(Z[s])$. Hence $C_N(G) \subset Z$ by (i) since $C_N(G) \lhd N \lhd U$. If G is Abelian then $G \subset C_N(G)$, so (iii) follows trivially.

The question of solvability when R has characteristic 2 and is 4-dimensional over Z, as well as for any group in a normal chain is open. We suspect that the groups in question fail to be solvable with suitable assumptions on card Z. Clearly, our present methods are not suitable for these considerable more complex cases.

7. One possible generalization of our results might be the case where we assume that R is semi-prime, that is R has no nilpotent ideals, instead of R simple. More seems to be required in this case than the assumption of idempotents. For example, let R be simple with $1 = e_1 + e_2 + e_3$, with the e_i orthogonal idempotents. Let $S = R_1 + R_2 + R_3$ where $R_i = e_i R e_i$. Then R_1 is a subalgebra of S and is invariant with respect to all inner automorphisms of S. Hence we cannot expect Theorem 4.5 to hold for semi-prime rings with idempotents. A similar example serves as a counter-example to the non-solvability of a noncentral normal subgroup in the case of one idempotent. Let $GF(3)_2$ be the 2×2 matrix ring over the field of three elements, and let S be any semi-prime ring with identity 1_s, and a nontrivial group of units. Then if $R = GF(3)_2 \oplus S$, where the sum is formally direct, then the group of units of $GF(3)_2$, considered in R, is a noncentral normal subgroup of R which is solvable. Thus it would seem that we must somehow eliminate the possibility that R is the direct sum of *bad* proper ideals.

DEFINITION. 7.1. R is a CSP ring if:

(i) *R has no nonzero nilpotent ideals.*

(ii) *R contains $e \neq 0, 1$ with $e^2 = e$, such that $eR(1-e)Re = eRe$ and $(1-e)ReR(1-e) = (1-e)R(1-e)$.*

We note that given any semi-prime ring S we can obtain CSP rings by considering the complete $n \times n$ matrix ring over S, for any $n \geqslant 2$.

LEMMA 7.2. *Let R be a* CSP *ring and suppose there is $z \in Z$ with $z^2 - z$ not a zero divisor in R. Then if N is an Abelian normal subgroup of U, $N \subset Z$.*

Proof. Let S be the ring generated by N over Z That is the $\{z_1 n_1 + \cdots + z_k n_k \mid z_i \in Z \text{ and } n_i \in N\}$. S is commutative. Since $N \lhd U$, if $a^2 = 0$, we have $(1-a)s(1+a) = s - as + sa - asa \in S$ for $s \in N$. As S is a subspace and $s \in N$ we get $z(sa - as) - zasa$ in S. Also, $(1 - za)s(1 + za) \in S$, so we obtain $z(sa - as) - z^2 asa$ in S. Hence $(z^2 - z)\, asa \in S$, where we can assume that $z^2 - z$ is not a divisor of zero. Since S is commutative, $(z^2 - z)\, asas = s(z^2 - z)\, asa$, and so $(z^2 - z)(asas - sasa) = 0$. This implies $asas = sasa$. Thus $asasa = sasa^2 = 0$. By Lemma 6.2 s commutes with $eR(1-e)$ and $(1-e)\,Re$ for the idempotent e in Definition 7.1. But by (ii) of Definition 7.1 it then follows that $s \in Z$. Hence $N \subset Z$.

DEFINITION. 7.3. R is called separating if there are $k, t, w \in Z$ with none of $k^2 - k$, $t^3 - t$, $w^4 - w$ zero divisors.

LEMMA 7.4. *Let R be a separating ring. If $N \lhd U$ with the derived group $N' \subset Z$, then for any $s \in N$ and $a^2 = 0$, the following relations hold*:

(i) $g_1(a) = 0 = s^2 a + 3as^2 - 3sas - s^{-1}as^3$

(ii) $$g_2(a) = 0 = asas - 2sasa + as^2 a - sas^{-1}as^2 + s^{-1}asas^2 - s^{-1}as^2 as + as^{-1}as^3$$

(iii) $$g_3(a) = 0 = s^{-1}asasas - as^{-1}asas^2 + as^{-1}as^2 as - asasa + sas^{-1}asas - sas^{-1}as^2 a$$

(iv) $g_4(a) = 0 = sas^{-1}asasa - as^{-1}asasas.$

Proof. Since $N \lhd U$, for $a^2 = 0$, we have $(1-a)s(1+a)$ in N, for $s \in N$. Thus $(s, (1-a)s(1+a)) \in Z$ and if this commutator is y, then $[s, y] = 0$. That is,

$$ss^{-1}(1-a)\,s^{-1}(1+a)s(1-a)s(1+a)$$
$$- s^{-1}(1-a)\,s^{-1}(1+a)s(1-a)s(1+a)s = 0.$$

Simplifying, we get $g_1(a) + g_2(a) + g_3(a) + g_4(a) = 0$. Note that we could have started with za in place of "a", for $z \in Z$, and obtained $g_1(za) + g_2(za) + g_3(za) + g_4(za) = 0$. Also we have $g_i(za) = z^i g_i(a)$. Hence we can obtain both $k^2(g_1(a) + \cdots + g_4(a)) = 0$ and $g_1(ka) + \cdots + g_4(ka) = 0$, where k is from Definition 7.3. Subtracting we get

$$(k^2 - k)\,g_1(a) + (k^2 - k^3)\,g_3(a) + (k^2 - k^4)\,g_4(a) = 0.$$

In a similar manner, using the t and w in Definition 7.3, we can eliminate the $g_3(a)$ and $g_4(a)$ terms to obtain $(k^2 - k)(t^3 - t)(w^4 - w)(g_1(a)) = 0$. It follows that $g_1(a) = 0$. We continue in a similar manner on $g_2(a) + g_3(a) + g_4(a) = 0$ and get $g_i(a) = 0$. To do this we need only realize that, for example, if $k^2 - k$ is not a zero divisor, neither is k, and so, $k^3 - k^2$ cannot be a zero divisor. Thus we have a sufficient number of suitable nonzero divisors in Z to enable us to obtain $g_i(a) = 0$.

THEOREM 7.5. *Let be a separating* CSP *ring such that if* $3r = 0$ *then* $r = 0$. *If* $N \lhd U$, $N \not\subset Z$, *then* N *is not solvable.*

Proof. If N were solvable we could consider its derived series $N \supset N' \supset \cdots \supset N^{(k)} \supset 1$. By Lemma 7.2 $N^{(k)} \subset Z$. We will have a contradiction and be done if $N^{(k-1)} \subset Z$. Now the derived group of $N^{(k-1)}$ is $N^{(k)} \subset Z$, so by Lemma 7.4 the relations $g_i(a) = 0$ hold for $s \in N^{(k-1)}$.

As $g_1(a) = 0$, we have $0 = asg_1(a)a = 3(asas^2a - as^2asa)$. Since $3r = 0$ implies $r = 0$, $asas^2a - as^2asa = 0$. Also,

$$0 = g_3(a)a = s^{-1}asasasa - as^{-1}(asas^2a - as^2asa) + sas^{-1}asasa.$$

Hence, $s^{-1}asasasa + sas^{-1}asasa = 0$. Thus

$$0 = g_4(a) = sas^{-1}asasa - as^{-1}asasas = -s^{-1}asasasa - as^{-1}asasas.$$

Left multiplication by s gives $-asasasa - sas^{-1}asasas = 0$. Using the relation obtained from $g_3(a)$ again, we have $-asasasa + s^{-1}asasasas = 0$. Thus $(sa)^4 = (as)^4$ and so $(as)^4a = 0$. Invoking Lemma 6.2 and using the same argument as at the end of Lemma 7.2 we get $N^{(k-1)} \subset Z$.

LEMMA 7.6. *Let* R *be a separating* CSP *ring. If* N *is a solvable normal subgroup of* U, *then* $3N \subset Z$.

Proof. Let $K = \{r \in R \mid 3r = 0\}$. One can easily verify that K is an ideal of R and that if $K \neq R$ then the quotient ring $R' = R/K$ is semi-prime and satisfies $3x' = 0$ implies $x' = 0$. Note that if $K = R$ then $3R = 0$ and so $3N = 0 \subset Z$. Hence we can assume that $K \neq R$. We claim that R' is separating. In fact, let x be any element of R which is not a zero divisor. Then x' is not a zero divisor in R'. To see this, suppose that $x'y' = 0$. Then $3xy = 0$, and so, $x(3y) = 0$. This implies that $y \in K$, or $y' = 0$.

We now have that R' satisfies the hypothesis of Theorem 7.5. Hence the image of N in R', being solvable and normal, must lie in the center of R'. What this says is given $s \in N$ and $r \in R$, $s'r' - r's' = 0$. In other words, $3sr - r3s = 0$, or $3N \subset Z$.

THEOREM 7.7. *Let R be a separating* CSP *ring. If $N \lhd U$, $N \not\subset Z$ then N is not solvable.*

Proof. If N were solvable it would have the derived series

$$N \supset N' \supset \cdots \supset N^{(k)} \supset 1.$$

By Lemma 7.2 $N^{(k)} \subset Z$. As in Theorem 7.5 we only need $N^{(k-1)} \subset Z$. Since the derived group of $N^{(k-1)}$ is in Z, the relations of Lemma 7.4 hold for the elements of $N^{(k-1)}$.

By Lemma 7.6 we have $3N \subset Z$ and so $g_1(a)$ in Lemma 7.4 becomes $s^2a - s^{-1}as^3 = 0$. This implies $s^3a = as^3$, for $s \in N^{(k-1)}$ and $a^2 = 0$. Now $0 = h(a) = ag_2(a)a = as^{-1}asas^2a - asas^{-1}as^2a - as^{-1}as^2asa$ and

$$0 = k(a) = g_3(a)a = s^{-1}asasasa - (as^{-1}asas^2a - as^{-1}as^2asa) + sas^{-1}asasa.$$

Using $h(a)$ in $k(a)$ yields $s^{-1}asasasa - asas^{-1}as^2a + sas^{-1}asasa = 0$. Multiply this last relation on the left by as^2, using $as^3 = s^3a$ and $a^2 = 0$. We get $asasasasa - as^2asas^{-1}as^2a = 0$. Since $s = s^{-2}s^3$, we get

$$asasasasa - as^2as^{-2}as^2as^2a = 0.$$

But $ag_4(a) = 0$ implies that $asas^{-1}asasa = 0$, so in particular,

$$as^2as^{-2}as^2as^2a = 0.$$

Hence $(as)^4a = 0$ and we proceed exactly as in the end of Theorem 7.2 or Theorem 7.5, using Lemma 6.2.

8. If we put somewhat stronger assumptions than having no nonzero nilpotent ideals on R, then to get Theorem 7.7 we need only assume that R contains some nonidentity idempotemt instead of assuming (ii) in Definition 7.1. We note that (ii) of Definition 7.1 arises in Lemma 7.2 and

Theorems 7.5 and 7.7 only after we have used Lemma 6.2 to get some element commuting with all $eR(1 - e)$ for e an idempotent. The problem is to show that such an element is in Z. We begin with

DEFINITION. 8.1. R is said to be prime if given a nonzero ideal J of R then $rJ = 0$ implies that $r = 0$.

Since in a prime ring no central element except zero is a divisor of zero, we note that for a prime ring to be separating it is sufficient that there exist elements, k, t, w in Z with $k^2 - k \neq 0$, $t^3 - t \neq 0$, and $w^4 - w \neq 0$.

LEMMA 8.2. *Let R be a prime ring containing a nonidentity idempotent. Let E be the subring of R generated by all the idempotents of R. Then E contains a nonzero ideal of R.*

Proof. We claim that E is a Lie ideal of R. It is enough to show that if $e_1, \ldots, e_n$ are idempotents and $r \in R$, then $[e_1 e_2 \cdots e_n, r]$ is in E. We proceed by induction on n. Note first the identity $[ab, c] = a[b, c] + [a, c]b$. Now $f_1 = e + ex(1 - e)$ and $f_2 = e + (1 - e)\ xe$ are idempotents and $f_1 - f_2 = [e, x]$. Suppose $[e_1 e_2 \cdots e_{n-1}, x] \in E$. Then clearly $[e_1 e_2 \cdots e_n, x] = e_1 e_2 \cdots e_{n-1}[e_n, x] + [e_1 \cdots e_{n-1}, x]e_n$ is in E, using our inductive assumption. Hence E is a Lie ideal and subring of R. This implies that E contains a nonzero ideal J of R, provided E is not commutative as a ring [3, p. 4]. But E is not commutative, since otherwise, if e is a nonidentity idempotent in R, then $[e, e - er(1 - e)] = 0$. This implies $er(1 - e) = 0$ for $r \in R$, and so $eR(1 - e) = 0$ which is impossible in a prime ring.

LEMMA 8.3. *Let R be prime with a nonidentity idempotent. If $[s, E] = 0$, where E is the ring generated by the idempotents in R, then $s \in Z$.*

Proof. By Lemma 8.2 E contains a nonzero ideal J of R. Since s centralizes E, s centralizes J. For $r \in R, j \in J$ we have $s(rj) = (rj)s = rsj$. Thus $[s, r]j = 0$, and so, $[s, R]J = 0$. As R is prime, $[s, R] = 0$. That is $s \in Z$.

LEMMA 8.4. *Let R be a separating prime ring with nonidentity idempotent. If N is a normal Abelian subgroup of U then $N \subset Z$.*

Proof. The proof of Lemma 7.2 holds exactly, through the point at which we apply Lemma 6.2. We note that the proof of Lemma 6.2 shows that the s with $(as)^k a = 0$ commutes with all idempotents of R. So we have that $s \in N$ commutes with E, the ring generated by the idempotents of R. By Lemma 8.3, $s \in Z$, and so $N \subset Z$.

THEOREM 8.5. *Let R be a separating prime ring with nonidentity idempotent. If $N \lhd U$, $N \not\subset Z$, the N is not solvable.*

Proof. Once again, if N were solvable we could consider its derived series $N \supset N' \supset \cdots \supset N^{(k)} \supset 1$. By Lemma 8.4, $N^{(k)} \subset Z$. Again we need only show $N^{(k-1)} \subset Z$. Now as R is prime it has a well-defined characteristic, hence either $3r = 0$ implies $r = 0$ or $3R = 0$. Corresponding to these cases we can use the proof of Theorem 7.5 or Theorem 7.7 to get $(as)^k a = 0$ for $s \in N^{(k-1)}$, $a^2 = 0$, and k a fixed integer. We now follow the argument in Lemma 8.4 to get $s \in Z$, and so $N^{(k-1)} \subset Z$.

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